GRAPH COVERINGS AND HARMONIC MAPS IN EXERCISES

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Abstract. The aim of notes is to give a background for the theory of graph coverings and harmonic maps. The basic definitions and main results of the theory are followed by exercises. Some of these excises are elementary, some of them will require non-trivial effort and some of them are unsolved problems. Also, the basic theory will be provided by numerous examples and the most exercises by solutions.

1. Graph coverings and covering groups

1.1. **Graph coverings and covering groups.** Let X and Y be connected graphs. A surjective morphism $\varphi : X \to Y$ is called a *(graph) covering* if for any vertex $x \in V(X)$ the restriction $\varphi|_{\mathrm{St}_X(x)} : \mathrm{St}_X(x) \to \mathrm{St}_Y(\varphi(x))$ is an isomorphism. *A covering group* of φ is defined as

$$Cov(\varphi) = \{h \in \mathrm{Aut}\,(X) : \varphi = \varphi \circ h\}.$$

The covering φ is called *regular* if $Cov(\varphi)$ act transitively on each fibre of φ and *irregular* othetwise. If $\varphi : X \to Y$ is a regular covering then $Y \cong X/Cov(\varphi)$. A finite sheeted covering $\varphi : X \to Y$ is regular if and only if the order of covering group $|Cov(\varphi)|$ coincides with the number of sheets of the covering.

If $\varphi : X \to Y$ is a covering and $\varphi(x) = y$ then there is a natural imbedding of the fundamental groups $\varphi_* : \pi_1(X, x) \to \pi_1(Y, y)$ induced by φ. Moreover, the index of subgroup $\varphi_*\pi_1(X, x)$ in $\pi_1(Y, y)$ coincides with the number of sheets of the covering. The covering φ is regular if and only if $\varphi_*\pi_1(X, x)$ is a normal subgroup in $\pi_1(Y, y)$. In the latter case, $Cov(\varphi)$ is canonnically isomorphic to the factor-group $\pi_1(Y, y)/\varphi_*\pi_1(X, x)$.

The coverings $\varphi : X \to Y$ and $\varphi' : X' \to Y$ are said to be *equivalent* if there is an isomorphism $h : X \to X'$ such that $\varphi = \varphi' \circ h$. The

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coverings $\varphi : X \to Y$ and $\varphi' : X' \to Y$ are equivalent if and only if the corresponding subgroups $\varphi_* \pi_1(X, x)$ and $\varphi'_* \pi_1(X', x')$ are conjugate in $\pi_1(Y, y)$.

1.1.1. *Coverings and transitive homomorphisms.* Let $\Gamma = \pi_1(X, x)$ be the fundamental group of a graph X at vertex x. It is well known that there is a one-to-one correspondence between the classes of equivalent n-fold coverings of X and the equivalence classes of transitive homomorphisms from Γ to the symmetric group $\mathbb{S}_n$ on n symbols. Recall that a homomorphism to $\mathbb{S}_n$ is called *transitive* if its image is a transitive subgroup in $\mathbb{S}_n$. Two homomorphisms, $\theta, \theta' : \Gamma \to \mathbb{S}_n$ are said to be equivalent if there exists $h \in \mathbb{S}_n$ such that $\theta' = h\,\theta\,h^{-1}$. [A. Hatcher, Algebraic Topology, Cambridge Univ. Press, Cambridge, 2002, p. 68].

Let X be a graph of genus g. Then Γ is a free group of rank g. Suppose that Γ is freely generated by the elements $x_1, x_2, \ldots, x_g$. Then an arbitrary transitive homomorphism $\theta : \Gamma \to \mathbb{S}_n$ is uniquely determined by the following conditions:

(i) $\theta(x_i) = \xi_i$, where $\xi_i \in \mathbb{S}_n$ for $i = 1, 2, \ldots, g$.
(ii) $\xi_1, \xi_2, \ldots, \xi_g$ generate a transitive subgroup in $\mathbb{S}_n$.

Two homomorphisms defined by tuples $(\xi_1, \xi_2, \ldots, \xi_g)$ and $(\xi'_1, \xi'_2, \ldots, \xi'_g)$ are equivalent if and only if exists $h \in \mathbb{S}_n$ such that $\xi'_i = h\,\xi_i\,h^{-1}$ for all $i = 1, 2, \ldots, g$.

1.1.2. *Graph coverings and voltage assignments.* Permutation voltage assignments were introduced by Gross and Tucker [9]. Let X be a finite connected graph, possibly including multiple edges or loops. It is *directed* if each edge (even a loop) is provided by the two possible directions. Let $D(X)$ be the set of the directed edges of X (also known as *darts, arcs* and so on in the literature). A permutation voltage assignment of X with voltages in the symmetric group $\mathbb{S}_n$ of degree n is a function $\phi : D(X) \to \mathbb{S}_n$ such that inverse edges have inverse assignments. The pair $(D(X), \phi)$ is called a permutation voltage graph. The *(permutation) derived graph* X^ϕ derived from a permutation voltage assignment ϕ is defined as follows:

$$V(X^\phi) = V(X) \times \{1, \cdots, n\}, \text{ and } ((u, j), (v, k)) \in D(G^\phi)$$

if and only if $(u, v) \in D(G)$ and $k = \phi(u, v)(j)$. The natural projection $\pi : X^\phi \to X$ that is a function from $V(X^\phi)$ onto $V(X)$ which erases the second coordinates gives a graph covering. Gross and Tucker [9] showed that every covering of a given graph arises from some permutation voltage assignment in a symmetric group. Moreover, such a covering is connected if and only if $\phi(D(X))$ is a transitive subgroup in $\mathbb{S}_n$.

1.1.3. *Regular coverings and ordinary voltage assignments.* Ordinary voltage assignments were introduced by Gross [8]. Let G be a finite group. Then a mapping $\omega : D(X) \to G$ is called an ordinary voltage assignment if $\omega(v,u) = \omega(u,v)^{-1}$ for each $(u,v) \in D(X)$. The *(ordinary) derived graph* X^ω derived from an ordinary voltage assignment ω is defined as follows: $V(X^\omega) = V(X) \times G$, and $((u,j),(v,k)) \in D(X^\omega)$ if and only if $(u,v) \in D(X)$ and $k = \omega(u,v)j$. Consider the natural projection $\pi : X^\omega \to X$ that is a function from $V(X^\omega)$ onto $V(X)$ which erases the second coordinates. Then the map $\pi : X^\omega \to X$ is a G-*covering* of X, that is a $|G|$-fold regular covering of X with the covering group G. Every regular covering of X can be obtained in such a way (see [9]).

1.1.4. *Reduced voltage assignments.* Let ω be an ordinary or a permutation voltage assignment on X. Choose a spanning tree T and a vertex v in X. For any edge e which is not in T there is an oriented closed walk w_e in X, based at v, with the following properties:

(a) $w_e - e \subseteq T$;
(b) the orientation of w_e agrees with that of e;
(c) w_e has the minimum length among all walks satisfying (a) and (b).

Putting $\omega'(e) = \omega(w_e)$, if e is not in T, and $\omega'(e) = 1$ otherwise, we obtain a new voltage assignment ω' called the (T,v)-reduction of ω.

Ezell's Theorem 4.1 in [12] implies that ω and ω' generate equivalent coverings.

Also, by Theorem 2 in [16] two ordinary voltage assignments $\psi : D(X) \to G$ and $\omega : D(X) \to G$ produce equivalent coverings if and only if their (T,v)-reductions ψ' and ω' are differ by an automorphism A of the group G such that $A \circ \psi' = \omega'$.

To obtain an analogue theorem for permutation voltage assignments, one has to replace the word "automorphism" by "inner automorphism" and "group G" by "symmetric group $\mathbb{S}_n$".

1.1.5. *Short way to construct coverings.* Let X be a graph of genus g. Choose a spanning tree T in X and g directed edges $e_1, e_2, \ldots, e_g$ from the compliment $X \setminus T$.

Using reduced permutation assignment. An arbitrary reduced permutation assignment $\psi : D(X) \to \mathbb{S}_n$ is uniquely determined by the following conditions:

(i) $\psi(e_i) = \xi_i$, where $\xi_i \in \mathbb{S}_n$ for $i = 1, 2, \ldots, g$ and $\psi(e) = 1$, for any edge e which is in T;
(ii) $\xi_1, \xi_2, \ldots, \xi_g$ generate a transitive subgroup in $\mathbb{S}_n$.

Then the permutation derived graph gives a required covering.

By (**??**) all connected n-fold coverings can be obtained in such a way. Two tuples $(\xi_1, \xi_2, \dots, \xi_g)$ and $(\xi'_1, \xi'_2, \dots, \xi'_g)$ give equivalent coverings if and only if exists $h \in \mathbb{S}_n$ such that $\xi'_i = h\,\xi_i\,h^{-1}$ for all $i = 1, 2, \dots, g$.

Using reduced ordinary assignment. An arbitrary reduced ordinary assignment $\omega : D(X) \to G$ is uniquely determined by the following conditions:

(i) $\omega(e_i) = a_i$, where $a_i \in G$ for $i = 1, 2, \dots, g$ and $\omega(e) = 1$, for any edge e which is in T;

(ii) $a_1, a_2, \dots, a_g$ generate group in G.

Then the ordinary derived graph gives a required G-covering.

By (**??**) all regular coverings can be obtained in such a way. Two tuples $(a_1, a_2, \dots, a_g)$ and $(a'_1, a'_2, \dots, a'_g)$ give equivalent G-coverings if and only if exists A of the group G such that $a'_i = A(a_i)$ for all $i = 1, 2, \dots, g$.

1.2. **Exercises.**

1.2.1. Draw all 2-fold coverings of the figure-eight graph. Show that all of them are regular.

1.2.2. Draw all 3-fold coverings of the figure-eight graph. How many of them are regular?

1.2.3. Show that composition $\psi \circ \varphi$ of two coverings $\varphi : X \to Y$ and $\psi : Y \to Z$ is a covering.

1.2.4. Let X be a connected graph and X is not a tree. Then G has infinitely many non-equivalent coverings.

1.2.5. Let X be a tree. Then up to equivalency X admits only one covering. Namely, the trivial covering $id : X \to X$. What about automorphisms of X? Are they also coverings?

1.2.6. Construct universal coverings for the following graphs:
(i) A loop (one vertex and one edge graph),
(ii) The figure eight graph,
(iii) Cyclic graph C_n.

1.2.7. Show that two cyclic graphs C_m and C_n share a finite sheeted covering.

1.2.8. Describe all coverings of a cyclic graph C_n.

1.2.9. Let $\varphi : X \to Y$ be a graph covering and Y is a tree. Then X is isomorphic to Y.

1.2.10. Let Y be a bipartite graph and $\varphi : X \to Y$ is a graph covering. Show that X is also a bipartite graph.

1.2.11. Let Y be a k-partite graph and $\varphi : X \to Y$ is a graph covering. Then X is also k-partite.

1.2.12. Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be regular graph coverings. Is it true that $\psi \circ \varphi : X \to Z$ is also regular graph covering?

1.2.13. Let $\varphi : X \to Y$ be an irregular graph covering. Prove that there exixst a regular graph covering $\psi : Z \to X$ such that the composition $\varphi \circ \psi : Z \to Y$ is also regular.

1.2.14. A regular covering $\varphi : X \to Y$ is called C_n-covering if the covering group of φ is isomorphic to cyclic group C_n of order n. Find the number of non-equivalent C_n-coverings of a given graph X of genus g. (Ph. Hall (1988), G. Jones (1975), A. D. Mednykh (1978))

Answer: $\#C_n$-coverings $= \sum\limits_{d|n} \mu(\frac{n}{d}) d^g$. This function is called Jordan g-function.

1.2.15. A covering $\varphi : X \to Y$ is called $\mathbb{D}_n$-covering if the covering group of φ is isomorphic to dihedral group $\mathbb{D}_n$ of order $2n$. Find the number of non-equivalent $\mathbb{D}_n$-coverings of a given graph X of genus g.

1.2.16. An n-fold covering $\varphi : X \to Y$ is called n-dihedral if there is a $\mathbb{D}_n$-covering $\psi : Z \to Y$ such that Z/C_2 is isomorphic to X, $D/\mathbb{D}_n$ is isomorphic to Y and $\varphi : X \cong Z/C_2 \to Y \cong Z/\mathbb{D}_n$ coincides with the covering induced by the group inclusion $C_2 < \mathbb{D}_n$. Find the number of non-equivalent n-dihedral covering of a graph Y of genus g.

Answer: unknown yet.

1.2.17. Find the number of non-equivalent n-fold bipartite coverings of a graph X of genus g. (Kwak, Lee, ...)

1.2.18. Let X and Y be graphs of genera g and h respectively ($g > h > 1$). Find an upper bound for the number of non-equivalent coverings $\varphi : X \to Y$.

Answer: unknown yet.

1.2.19. Two coverings $\varphi : X \to Y$ and $\psi : X \to Y$ are isomorphic if there are automorphisms $\alpha : Y \to Y$, $\beta : X \to X$ such that $\alpha \circ \varphi = \psi \circ \beta$. For given graphs X and Y of genera g and h respectively ($g > h > 1$) find an upper bound for the number of non-isomorphic coverings $\varphi : X \to Y$.

Answer: unknown yet.

2. Spanning trees and Laplacians

2.1. **Laplacian matrix and Laplacian spectrum.** The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have a physical interpretation in various physical and chemical theories. The related matrix the adjacency matrix of a graph and its eigenvalues were much more investigated in the past than the Laplacian matrix. In the same time, the Laplacian spectrum is much more natural and more important than the adjacency matrix spectrum because of it numerous application in mathematical physics, chemistry and financial mathematics.

The graphs in this section are unoriented, but they may have loops and multiple edges. We also allow weighted graphs which are viewed as a graph which has for each pair u, v of vertices, assigned a certain weight a_{uv}. The weights are usually real numbers and they must satisfy the following conditions:

(i) $a_{uv} = a_{vu}$, $v, u \in V(G)$,
(ii) $a_{vu} \neq 0$, if and only if v and u are adjacent in G.
(iii) $a_{uv} \geq 0$, $v, u \in V(G)$.

Unweighted graphs can be viewed as a special case of weighted graphs, by specifying, for each $u, v \in V(G)$, the weight a_{uv} to be equal to the number of edges between u and v. The matrix $A = A(G) = [a_{uv}]_{u,v\in V(G)}$, is called the *adjacency matrix* of the graph G.

Let $d(v)$ denote the degree of $v \in V(G)$, $d(v) = \sum_u a_{uv}$, and let $D = D(G)$ be the diagonal matrix indexed by $V(G)$ and with $d_{vv} = d(v)$. The matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. It should be noted that loops have no influence on $L(G)$. The matrix $L(G)$ is sometimes called the *Kirchhoff matrix* of G due to its role in the well-known Matrix-Tree Theorem (cf, exercise 4.44) which is usually attributed to Kirchhoff.

Throughout the paper we shall denote by $\mu(G, x)$ the characteristic polynomial of $L(G)$. Its roots will be called the Laplacian eigenvalues (or sometimes just eigenvalues) of G. They will be denoted by $\mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G)$, $(n = |V(G)|)$, always enumerated in increasing order and repeated according to their multiplicity.

2.1.1. *Spectrum of some graphs.*

1°. **The complete graph.** The Laplace spectrum of the complete graph K_n on n vertices is 0^1, n^{n-1}.

2°. **The complete bipartite graph.** The Laplace spectrum of the complete bipartite graph $K_{m,n}$ is 0^1, m^{n-1}, n^{m-1}, $(m+n)^1$.

3°. **The cycle graph.** The Laplace spectrum of the n-cycle graph C_n consists of the numbers $2 - 2\cos(2\pi j/n)$, $(j = 0, \ldots, n-1)$.

4°. **The path graph.** The Laplace spectrum of the path graph P_n with n vertices consists of the numbers $2 - 2\cos(\pi j/n)$, $(j = 0, \ldots, n-1)$.

2.1.2. *Further properties of the Laplacian spectrum.*

Fiedler [F1] derived the following result about the Cartesian products of graphs.

Theorem 1. *The Laplacian eigenvalues of the Cartesian product $X_1 \times X_2$ of graphs X_1 and X_2 are equal to all the possible sums of eigenvalues of the two factors:*

$$\lambda_i(X_1) + \lambda_j(X_2),\ i = 1, \ldots, |V(X_1)|,\ j = 1, \ldots, |V(X_2)|.$$

By applying Theorem 1 we can easily determine the spectrum of lattice graphs. The $m \times n$ lattice graph is just the Cartesian product of paths, $P_m \times P_n$. The spectrum of P_k is [AnM]

$$l_i^{(k)} = 4\sin^2\frac{\pi i}{2k}, i = 0, 1, \ldots, k-1.$$

So $P_m \times P_n$ has eigenvalues

$$\lambda_{i,j} = l_i^{(m)} + l_j^{(n)} = 4\sin^2\frac{\pi i}{2m} + 4\sin^2\frac{\pi j}{2n}, i = 0, 1, \ldots, m-1, j = 0, 1, \ldots, n-1.$$

Corollary 1. *[Kel'mans] Let $X_1 * X_2$ denote the join of X_1 and X_2, i.e. the graph obtained from the disjoint union of X_1 and X_2 by adding all possible edges uv, $u \in V(X_1)$, $v \in V(X_2)$. Then*

$$\mu(X_1 * X_2, x) = \frac{x(x - n_1 - n_2)}{(x - n_1)(x - n_2)}\mu(X_1, x - n_2)\mu(X_2, x - n_1).$$

where n_1 and n_2 are orders of X_1 and X_2, respectively and $\mu(X, x)$ is the characteristic polynomial of the Kirchhoff matrix of X.

A generalization of the Matrix-Tree-Theorem was obtained by Kelmans [K3] who gave a combinatorial interpretation to all the coefficients of $\mu(X, x)$ in terms of the numbers of certain subforests of the graph. This result has been obtained even in greater generality (for weighted graphs) by Fiedler and Sedláček [FS].

Theorem 2. *[FS, K3] If $\mu(X, x) = x^n + c_1 x^{n-1} + \ldots + c_{n-1} x$ then*

$$c_i = (-1)^i \sum_{S \subset V, |S| = n-i} t(X_S),$$

where $t(H)$ is the number of spanning trees of H, and X_S is obtained from X by identifying all points of S to a single point.

2.2. **Some properties of Chebyshev polynimials.**

The Chebyshev polynimial of the first kind is defined by the formula

$$T_n(x) = \cos(n \arccos x).$$

Eqivalently,

$$T_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}.$$

Also, $T_n(x)$ satisfies the recursive relation

$$T_0(x) = 1,\, T_1(x) = x,\, T_n(x) = 2x \cdot T_{n-1}(x) - T_{n-2}(x) = 1,\, n \geq 2.$$

The Chebyshev polynimial of the second kind is defined by the formula

$$U_n(x) = \frac{\sin((n + 1) \arccos x)}{\sin(\arccos x)}.$$

Eqivalently,

$$U_n(x) = \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{2\sqrt{x^2 - 1}}.$$

Also, $U_n(x)$ satisfies the recursive relation

$$U_0(x) = 1,\, U_1(x) = 2x,\, U_n(x) = 2x \cdot U_{n-1}(x) - U_{n-2}(x) = 1,\, n \geq 2.$$

We have $U_n(\cos \frac{k\pi}{n+1}) = 0, k = 1, 2, \ldots , n$. Hence

$$U_n(x) = 2^n \prod_{k=1}^{n} (x - \cos \frac{k\pi}{n + 1}).$$

Since

$$U_n(x) = (-1)^n U_n(-x) = 2^n \prod_{k=1}^{n} (x + \cos \frac{k\pi}{n + 1})$$

we obtain

$$U_n^2(x) = \prod_{k=1}^{n} (4x^2 - 4 \cos^2 \frac{k\pi}{n + 1}).$$

Polynomials $T_n(x)$ and $U_{n-1}(x)$ are related by the following identity

$$T_n^2(x) + (x^2 - 1)U_{n-1}^2(x) = 1.$$

2.3. **Spanning trees.**

2.3.1. A *spanning tree* T of a connected, undirected graph G is a tree composed of all the vertices and some (or perhaps all) of the edges of G. In other words, a spanning tree of G is a selection of edges of G that form a tree spanning every vertex. That is, every vertex lies in the tree, but no cycles (or loops) are formed. On the other hand, every bridge of G must belong to T. A spanning tree of a connected graph G can also be defined as a maximal set of edges of G that contains no cycle, or as a minimal set of edges that connect all vertices.

2.3.2. *Counting spanning trees.* The number $t(G)$ of spanning trees of a connected graph is a well-studied invariant. In some cases, it is easy to calculate $t(G)$ directly. For example, if G is itself a tree, then $t(G) = 1$, while if G is the cycle graph C_n with n vertices, then $t(G) = n$. For any graph G, the number $t(G)$ can be calculated using Kirchhoff's matrix-tree theorem.

Cayley's formula for the number of spanning trees in the *complete graph* K_n with n vertices states that $t(K_n) = n^{n-2}$. If G is the *complete bipartite graph* $K_{m,n}$, then $t(G) = m^{n-1}n^{m-1}$, while if G is the n-dimensional *hypercube graph* Q_n, then $t(G) = 2^{2^n-n-1}\prod\limits_{k=2}^{n} k^{\binom{n}{k}}$. These formulae are also consequences of the matrix-tree theorem.

2.4. **Exersises.**

2.1.1. Draw all spanning trees for graph K_4.

2.1.2. Find set of spanning trees for the cube and for the octahedral graph. Is there a natural one-to-one correspondence between these two sets?

2.1.3. Find the number t_n of spanning trees for the following graphs

(i) Path graph P_n,
(ii) Cyclic graph C_n,
(iii) Complete graph K_n,
(iv) Ladder graph $L_n = P_2 \times P_n$,
(v) Fan graph $F_n = P_n + K_1$.

2.1.4. Find the number of spanning trees of a graph G through the spectrum of the Laplacian of G.

2.1.5. Simplify the above formula for k-regular graphs.

2.1.6. Find $t(\dot{G})$, where $\dot{G}$ is the join of a graph G and a point graph.

2.1.7. Let $\varphi : G \to H$ be a graph covering. Prove that $t(H)|t(G)$.

2.1.8. Find the number of spanning trees for the direct product of graphs G and H. There are four definition of direct product of graphs. The spectrums of G and H are known.

2.1.9. Find the number of spanning trees for

(i) Ladder $P_2 \times P_n$,
(ii) Rectangular $P_n \times P_m$,
(iii) Cube $P_2 \times P_2 \times P_2$,
(iv) Parallelepiped $P_k \times P_l \times P_m$,
(v) Torus $C_n \times C_m$.

2.1.10. Find the number of spanning trees $t(W_n)$ for the wheel graph $W_n = C_n + K_1$.

2.1.11. For any graph X we denote by $t(X)$ the total number of spanning trees of X. Let $A = A(X)$ denote the adjacency matrix of X and D be the diagonal matrix of degrees of X. Then the *Kirchhoff matrix* is defined as $H = D - A$.

Prove the celebrated Kirchhoff Matrix Tree Theorem [??]: All cofactors of H are equal to $t(X)$.

2.1.12. Let $0 = \mu_1 \le \mu_2 \le \cdots \le \mu_n$ denote the eigenvalues of the Kirchhoff matrix H of a n point graph. Prove the following result obtained by by Kel'mans and Chelnokov [KelChel]:

$$t(X) = \frac{1}{n}\prod_{k=2}^{n}\mu_k.$$

2.1.13. Prove the following Temperley's formula [Temp]

$$t(X) = \frac{1}{n^2}\det(H + J),$$

where J is the $n \times n$ matrix all of whose elements are unity.

2.1.14. Let X be a regular graph of degree r. Then [Sachs]

$$t(X) = \frac{1}{n}\prod_{k=2}^{n}(r - \lambda_k),$$

where $\lambda_n \le \cdots \le \lambda_2 \le \lambda_1 = r$ the eigenvalues of the adjacency matrix A.

2.1.15. Let e an edge of a graph X. Show that the number of spanning trees of X which contain e is $t(X/e)$, where $t(X/e)$ denotes the graph obtained by coalescing the endpoints of the edge e. Show the result, which is apparently due to Feussner [Feu] (see also Moon [Moon])

$$t(X) = t(X - e) + t(X/e).$$

2.1.16. Let X_s denotes the graph that results from subdividing an edge e of a graph X. Then

$$t(X_s) = t(X/e) + 2t(X - e) = t(X) + t(X - e).$$

2.1.17. Let X_p denotes the results of adding an edge in parallel an edge e of a graph X. Then

$$t(X_p) = t(X) + t(X/e).$$

2.1.17*. Let X be a finite connected graph. Suppose that there is an edge e of X such that the complement $X\backslash e$ consists of two connected graphs X_1 and X_2. Prove that

$$t(X) = t(X_1)\, t(X_2).$$

2.1.18. Prove that the number of spanning trees for the prism $P_2 \times C_n$ is given by the formula

$$t(P_2 \times C_n) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n - 2).$$

2.1.19. Prove that the number of spanning trees for the Moebius ladder M_n is given by the formula

$$t(M_n) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n + 2).$$

2.1.20. Prove the following result obtained by Boesch and Prodinger [BoProd]: the number of spanning trees for the complete prism $K_m \times C_n$ is given by the formula

$$t(K_m \times C_n) = \frac{n}{m}\, 2^{m-1}\big[(T_n(1+\frac{m}{2}) - 1)\big]^{m-1},$$

where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of the first kind.

2.1.20*. Prove that the number of spanning trees for the lattice graph $L_{m,n} = K_m \times K_n$ is given by the formula

$$t(L_{m,n}) = m^{m-1} n^{n-1} (m+n)^{m+n-1}.$$

2.1.21. Let X be a graph on m vertices with the Laplacian eigenvalues $0 = \mu_1(X) \le \mu_2(X) \le \cdots \le \mu_m(X)$. Then [Chen Xiebin]

$$t(X \times P_n) = t(X) \prod_{i=2}^{m} U_{n-1}(1 + \frac{\mu_i(X)}{2}),$$

where $U_{n-1}(x) = \sin(n \arccos x)/\sin(\arccos x)$ is the Chebyshev polynomial of the second kind.

2.1.22. Let X be a graph on m vertices with the Laplacian eigenvalues $0 = \mu_1(X) \le \mu_1(X) \le \cdots \le \mu_m(X)$. Then [Chen Xiebin]

$$t(X \times C_n) = n\, t(X) \prod_{i=1}^{m} U_{n-1}^2(\frac{1}{2}\sqrt{4 + \mu_i(X)}).$$

2.1.23. Prove the following identities [Chen Xiebin]:

(i) $t(P_m \times P_n) = \prod\limits_{k=1}^{m-1} U_{n-1}(2 - \cos \frac{k\pi}{m});$

(ii) $t(C_m \times P_n) = \prod\limits_{k=1}^{m-1} U_{n-1}(2 - \cos \frac{2k\pi}{m});$

(iii) $t(C_m \times C_n) = mn \prod\limits_{k=1}^{m-1} U_{n-1}^2(\frac{1}{2}\sqrt{6 - 2\cos \frac{2k\pi}{m}}).$

2.5. **Solutions.**

2.1.11.– 2.1.13. Let L be the Kirchhoff-Laplace matrix with eigenvalues $0 = \mu_1 \le \mu_2 \le \ldots \le \mu_n$. Let l_{xy} be the (x, y)-cofactor of L. (The (i, j)-cofactor of a matrix M is by definition $(-1)^{i+j} \det M(i, j)$, where $M(i, j)$ is the matrix obtained from M by deleting row i and column j. Note that l_{xy} does not depend on an ordering of the vertices of X.)

We set $N = t(X)$ and show that

$$N = l_{xy} = \det(L + \frac{1}{n}J) = \frac{1}{n}\mu_2 \ldots \mu_n \text{ for any } x, y \in V(X).$$

Let L^S, for $S \subset V(X)$, denote the matrix obtained from L by deleting the rows and columns indexed by S, so that $l_{xx} = \det L^{\{x\}}$. The equality $N = l_{xx}$ follows by induction on n, and for fixed $n > 1$ on the number of edges incident with x. Indeed, if $n = 1$ then $l_{xx} = 1$. Otherwise, if x has degree 0, then $l_{xx} = 0$ since $L^{\{x\}}$ has zero row sums. Finally, if xy

is an edge, then deleting this edge from X diminishes l_{xx} by $\det L^{\{x,y\}}$, which by induction is the number of spanning trees of X with edge xy contracted, which is the number of spanning trees containing the edge xy. This shows $N = l_{xx}$.

Now $\mu(X,t) = \det(tI - L) = t\prod_{i=2}^{n}(t-\mu_i)$ and $(-1)^{n-1}\mu_2\dots\mu_n$ is the coefficient of t, that is, is $\frac{d}{dt}\det(tI-L)|_{t=0}$. But $\frac{d}{dt}\det(tI-L) = \sum_x \det(tI - L^{\{x\}})$, so $\mu_2\dots\mu_n = \sum_x l_{xx} = nN$.

Since the sum of the columns of L is zero, so that one column is minus the sum of the other columns, we have $l_{xx} = l_{xy}$ for any x, y. Finally, the eigenvalues of $L+\frac{1}{n}J$ are $\frac{1}{n}$ and $\mu_2,\dots,\mu_n$, so $\det(L+\frac{1}{n}J) = \frac{1}{n}\mu_2\dots\mu_n$.

Example 1. *The multigraph of valency k on two vertices has Laplace matrix $L = \{\{k,-k\},\{-k,k\}\}$ so $\mu_1 = 0, \mu_2 = 2k$, and $N = 1\cdot 2k = k$.*

Example 2. *Consider the complete graph K_n, then $\mu_2 = \ldots = \mu_n = n$, and therefore K_n has $N = n^{n-2}$ spanning trees. This formula is due to Cayley [Cayley].*

2.1.14. A graph X is called regular of degree (or valency) r when every vertex has precisely r neighbors. So, X is regular of degree r precisely when its adjacency matrix A has row sums r, i.e., when $A1 = r1$ (or $AJ = rJ$). If X is regular of degree r, then for every eigenvalue λ we have $|\lambda| \le r$. (One way to see this is by observing that if $|t| > r$ then the matrix $tI - A$ is strictly diagonally dominant, and hence nonsingular, so that t is not an eigenvalue of A.) If X is regular of degree r, then $L = rI - A$. It follows that if X has ordinary eigenvalues $r = \lambda_1 \ge \ldots \ge \lambda_n$ and Laplace eigenvalues $0 = \mu_1 \le \mu_2 \le \ldots \le \mu_n$, then $\mu_i = r - \lambda_i$ for $i = 1,\dots,n$ and by the previous solution $t(X) = \frac{1}{n}\,\mu_2\dots\mu_n = \frac{1}{n}(r-\lambda_2)\dots(r-\lambda_n)$.

2.1.20. The Laplace spectrum of the complete graph K_m with m vertices is $\mu_0 = 0$, $\mu_i = m$, $i = 1,\dots, m-1$. The graph C_n is regular of valency 2, so its Laplace spectrum consists of the numbers $\lambda_j = \lambda_j(C_n) = 2 - 2\cos(2\pi j/n)$, $j = 0,\dots, n-1$.

$t(K_m\times C_n) = \frac{1}{m\,n}\prod_{i=0}^{m-1}\prod_{j=0}^{n-1}(\mu_i+\lambda_j)$, where $i+j > 0$ $= \frac{1}{m\,n}\prod_{j=1}^{n-1}\lambda_j\prod_{i=1}^{m-1}\mu_i\prod_{i=1}^{m-1}\prod_{j=1}^{n-1}(\lambda_j+\mu_i) = t(C_n)t(K_m)\prod_{i=1}^{m-1}\prod_{j=1}^{n-1}(m+2-2\cos(2\pi j/n)) = n\,m^{m-2}(\prod_{j=1}^{n-1}(m+4-4\cos(\pi j/n)^2))^{m-1} = n\,m^{m-2}\left[U_{m-1}^2(\sqrt{\frac{m+4}{4}})\right]^{m-1}$.

Then the result follows from the identities

$$U^2_{m-1}(x) = \frac{1}{2(1-x^2)}(1 - T_{2m}(x)) = \frac{1}{2(1-x^2)}(1 - T_m(2x^2-1)).$$

2.1.20*. The Laplace spectrums of the graphs K_m and K_n are $\mu_0 = 0$, $\mu_i = m$, $i = 1, \dots, m-1$ and $\lambda_0 = 0$, $\lambda_j = n$, $j = 1, \dots, n-1$. Then $t(K_m \times K_n) = \frac{1}{mn}\prod_{i=0}^{m-1}\prod_{j=0}^{n-1}(\mu_i + \lambda_j)$, where $i + j > 0$ $= \frac{1}{mn}\prod_{j=1}^{n-1}\lambda_j \prod_{i=1}^{m-1}\mu_i \prod_{i=1}^{m-1}\prod_{j=1}^{n-1}(\lambda_j + \mu_i) = m^{m-2}n^{n-2}(m+n)^{m+n-2}$.

2.1.21. The Laplace spectrum of the path graph P_n with n vertices is $2 - 2\cos(\pi j/n)$, $j = 0, 1, \dots, n-1$. Hence

$t(X \times P_n) = \frac{1}{mn}\prod_{i=1}^{m}\prod_{j=0}^{n-1}(\mu_i(X) + 2 - 2\cos(\pi j/n))$, where $i + j > 1$ $= \frac{1}{mn}\prod_{i=2}^{m}\mu_i(X)\prod_{j=1}^{n-1}(2 - 2\cos(\pi j/n))\prod_{i=1}^{m-1}\prod_{j=1}^{n-1}(\mu_i(X) + 2 - 2\cos(\pi j/n)) = t(X)t(P_n)\prod_{i=1}^{m-1}\prod_{j=1}^{n-1}(\mu_i(X) + 2 - 2\cos(\pi j/n)) = t(X)\prod_{i=1}^{m-1}U_{n-1}(1 + \frac{\mu_i(X)}{2})$.

2.1.22. The Laplace spectrum of the circle graph C_n with n vertices is $2 - 2\cos(2\pi j/n)$, $j = 0, 1, \dots, n-1$. Hence

$t(X \times C_n) = \frac{1}{mn}\prod_{i=1}^{m}\prod_{j=0}^{n-1}(\mu_i(X) + 2 - 2\cos(2\pi j/n))$, where $i + j > 1$ $= \frac{1}{mn}\prod_{i=2}^{m}\mu_i(X)\prod_{j=1}^{n-1}(2 - 2\cos(2\pi j/n))\prod_{i=1}^{m-1}\prod_{j=1}^{n-1}(\mu_i(X) + 2 - 2\cos(2\pi j/n)) = t(X)t(C_n)\prod_{i=1}^{m-1}\prod_{j=1}^{n-1}(4 + \mu_i(X) - 4\cos^2\frac{\pi j}{n}) = n\, t(X)\prod_{i=1}^{m-1}U^2_{n-1}(\frac{1}{2}\sqrt{4 + \mu_i(X)})$.

3. Harmonic maps and harmonic actions

3.1.1. Show that the natural C_n−covering of the wheel graphs $W_{nk} \to W_k$ is a harmonic map. Check that C_n acts pure harmonically on W_{nk}.

3.1.2. Show that "zig-zag" map of P_3 onto P_2 is a harmonic map. Find branched points of this map.

3.1.3. Construct a harmonic map of tree onto a tree with one branched point of order n.

3.1.4. Describe all harmonic maps between trees.
Answer: unknown yet.

3.1.5. Let group G acts purely harmonically on a graph X. Then the factor map $X \to X/G$ is harmonic map.

3.1.6. Construct a C_6-regular harmonic map of a graph $K_{2,3}$ onto a segment P_2.

3.1.7. Let a finite group G acts on a graph X fixing only one edge e. Replace e by $|G|$ parallel edges to get graph X'. Then G acts harmonically on X'.

3.1.8. Construct a $3-$fold uniform harmonic map that is irregular.

3.1.9. Define the monodromy group of a harmonic map. Show that a harmonic map is regular if and only if its monodromy group is regular. (when the notion of monodromy group is well-defined?)

4. Jacobians

4.1. **Basic definitions.** The notion of the Jacobian group of graph (also known as the Picard group, critical group, sandpile group, dollar group) was independently given by many authors ([7], [3], [4], [2]). This is a very important algebraic invariant of a finite graph. In particular, the order of the Picard group coincides with the number of spanning trees for a graph. Following Baker-Norine [3] we define the the Jacobian group of a graph as follows.

Let G be a finite, connected multigraph without loops. Let $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. Denote by $Div(G)$ a free Abelian group on $V(G)$. We refer to elements of $Div(G)$ as divisors on G. Each element $D \in Div(G)$ can be uniquely presented as $D = \sum_{x \in V(G)} D(x)(x)$, $D(x) \in \mathbb{Z}$. We define the degree of D by the formula $deg(D) = \sum_{x \in V(G)} D(x)$. Denote by $Div^0(G)$ the subgroup of $Div(G)$ consisting of divisors of degree zero.

Let f be a $\mathbb{Z}$-valued function on $V(G)$. We define the divisor of f by the formula

$$div(f) = \sum_{x \in V(G)} \sum_{xy \in E(G)} (f(x) - f(y))(x).$$

The divisor $div(f)$ can be naturally identified with the graph-theoretic Laplacian Δf of f. Divisors of the form $div(f)$, where f is a $\mathbb{Z}$-valued function on $V(G)$, are called principal divisors. Denote by $Prin(G)$ the group of principal divisors of G. It is easy to see that every principal divisor has a degree zero, so that $Prin(G)$ is a subgroup of $Div^0(G)$.

The *Jacobian group* (or *Jacobian*) of G is defined to be quotient group

$$Jac(G) = Div^0(G)/Prin(G).$$

By making use of the Kirchhoff Matrix-Tree theorem [13] one can show that $Jac(G)$ is a finite Abelian group of order $t(G)$, where $t(G)$ is number of spanning trees of G. An arbitrary finite Abelian group is the Jacobian group of some graph.

4.2. **Abel-Jacobi map.** For a fixed base point $x_0 \in V(G)$ we define the *Abel-Jacobi map* $S_{x_0} : G \to Jac(G)$ by the formula $S_{x_0}(x) = [(x) - (x_0)]$, where $[d]$ is an equivalence class of divisor d. If graph G is 2-edge connected (=bridgeless) then S_{x_0} is an imbedding [3].

4.3. **Jacobians and flows.** We endow each edge of G by two possible orientations. Since G has no loops it is well-defined procedure. Let $\vec{E} = \vec{E}(G)$ be the set of oriented edges of G. For $e \in \vec{E}$ we denote initial vertex $o(e)$ and terminus vertex $t(e)$, respectively. We define the flow of e by the formula $\omega(e) = [t(e) - o(e)]$. We note that

$$\omega(e) = [[t(e) - x_0] - [o(e) - x_0]] = S_{x_0}(t(e)) - S_{x_0}(o(e))$$

does not depend of the choice of initial point x_0. By virtue of Lemma 1.8 in [3] (see also [2]) the Jacobian $Jac(G)$ is an Abelian group generated by flows $\omega(e), e \in \vec{E}$, whose defining relations are given by the two following Kirchhoff's laws.

(K_1) The flow through each vertex of G is equal to zero. It means that

$$\sum_{e \in \vec{E}, t(e)=x} \omega(e) = 0 \text{ for all } x \in V(G).$$

(K_2) The flow along each closed orientable walk W in G is equal to zero. That is

$$\sum_{e \in W} \omega(e) = 0.$$

Recall that the closed orientable walk in G is a sequence of orientable edges $e_i \in \vec{E}(G)$, $i = 1, \ldots, n$ such that $t(e_i) = o(e_{i+1})$ for $i = 1, \ldots, n-1$ and $t(e_n) = o(e_1)$.

4.4. **Smith normal form.** Let $\mathcal{A}$ be a finite Abelian group generated by $x_1, x_2, \ldots, x_n$ and satisfying the system of relations

$$\sum_{j=1}^{n} a_{ij}x_j = 0,\ i = 1,\ \ldots,\ m,$$

where $A = \{a_{ij}\}$ is an integer $m \times n$ matrix. Set $d_j,\ j = 1,\ \ldots,\ r$, for the greatest common divisor of all $j \times j$ minors of A. Then,

$$\mathcal{A} \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2/d_1} \oplus \mathbb{Z}_{d_3/d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r/d_{r-1}}.$$

The latter decomposition is known as the Smith Normal Form. See ([24], Ch. 3.22) for details.

4.5. **Jacobians and Laplacians.** Consider the Laplacian matrix $L(G)$ as a homomorphism $\mathbb{Z}^{|V|} \to \mathbb{Z}^{|V|}$, where $|V| = |V(G)|$ is the number of vertices of G. Then $\mathrm{coker}(L(G)) = \mathbb{Z}^{|V|}/\mathrm{im(L(G))}$ is an abelian group. For $1 \le i \le |V|$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{|V|}$, be the i-th standard basis, and x_i be its image in coker(L(G)). It is known that $\mathrm{coker}(L(G))$ is determined by the generators $x_1, \ldots, x_{|V|}$ and the relations $(x_1, \ldots, x_{|V|})L(G) = 0$.

Two integral matrices A and B are equivalent (written $A \sim B$) if there are unimodular matrices P and Q such that $B = PAQ$. Equivalently, B is obtained from A by a sequence of elementary row and column operations: (1) the interchange of two rows or columns, (2) the multiplication of any row or column by -1, (3) the addition of any integer times of one row (resp. column) to another row (resp. column).

It is easy to see that $A \sim B$ implies that coker(A) $\sim$ coker(B). The *Smith normal form* is a diagonal canonical form for our equivalence relation: every $n \times n$ integral matrix A is equivalent to a unique diagonal matrix $\mathrm{diag}(s_1(A), \ldots, s_n(A))$, where $s_i(A)$ divides $s_{i+1}(A)$ for $i = 1, 2, \ldots, n-1$. The i-th diagonal entry of the Smith normal form of A is usually called the i-th invariant factor of A. We will use the fact that the values $s_i(A)$ can also be interpreted as follows: for each i, the product $s_1(A)s_2(A)\cdots s_i(A)$ is the greatest common divisor of all $i \times i$ minors of A.

The classification theorem for finitely generated abelian groups asserts that $\mathrm{coker}(L(G))$ has a direct sum decomposition

$$\mathrm{coker}(L(G)) \cong \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_{|V|}},$$

where the nonnegative integers t_i are the diagonal entries of the Smith normal form of the relation matrix $L(G)$, satisfying $t_i \big| t_{i+1}$, $(1 \le i \le$

$|V|$). Since G is connected, it is not hard to see that $L(G)$ has rank $|V|-1$, and the kernel of $L(G)$ is spanned by the vectors in $\mathbb{Z}^{|V|}$ which are constant on the vertices. It follows that $t_{|V|}=0$, $t_1\cdots t_{|V|-1}\neq 0$ and $\mathbb{Z}_{t_{|V|}}=\mathbb{Z}$. Now

$$\operatorname{coker}(L(G))=\mathbb{Z}^{|V|}/\operatorname{im}(L(G))\cong\mathbb{Z}\oplus\operatorname{Jac}(G),$$

where

$$\operatorname{Jac}(G)=\mathbb{Z}_{t_1}\oplus\mathbb{Z}_{t_2}\oplus\cdots\oplus\mathbb{Z}_{t_{|V|-1}}$$

is the Jacobian group of G.

4.6. **Exersises.**

4.6.1. Let $\varphi: X\to Y$ be a harmonic map of graphs. Show that induced Albanese functor $\varphi_*:\operatorname{Jac}(X)\to\operatorname{Jac}(Y)$ is an epimorphism.

4.6.2. Let $\phi: X\to Y$ be a harmonic map of graphs. Show that induced Picard functor $\varphi^*:\operatorname{Jac}(Y)\to\operatorname{Jac}(X)$ is injective.

4.6.3. Show that Albanese functor is covariant. That is $\varphi_*\circ\psi_*=(\varphi\circ\psi)^*$.

4.6.4. Show that Picard functor is contravariant. That is $\varphi^*\circ\psi^*=(\psi\circ\varphi)_*$.

4.6.5. Let $\varphi: X\to Y$ be a harmonic map of degree n. Show that $\varphi_*\circ\varphi^*=n\cdot\mathrm{Id}_Y$.

4.6.6. Let $\varphi: X\to X$ be a graph isomorphism. Show that φ_* and φ^* are automorphisms of the Jacobian $\operatorname{Jac}(X)$. Could φ be non-trivial automorphism if φ_* and φ^* are trivial automorphisms.

4.6.7. Let X be a finite 2-edge-connected graph. Show that the correspondence $\varphi\in\operatorname{Aut}(X)\to\varphi_*\in\operatorname{Jac}(X)$ is a group monomorphism.

4.6.8. Let X be a finite 2-edge-connected graph. Show that the correspondence $\varphi\in\operatorname{Aut}(X)\to\varphi^*\in\operatorname{Jac}(X)$ is a group monomorphism.

4.6.9. Find graph X such that $\operatorname{Aut}(X)\cong\operatorname{Jac}(X)$.

4.6.10. Let P_n be a path graph on $n+1$ vertices. Show that $\operatorname{Jac}(P_n)=0$.

4.6.11. Let C_n be a cyclic graph on n vertices. Show that $\operatorname{Jac}(C_n)=\mathbb{Z}_n$.

4.6.12. Let K_n be the complete graph on n vertices. Prove that $\mathrm{Jac}(K_n) = \mathbb{Z}_n^{n-2}$.

4.6.13. Let $K_{m,n}$ be the complete bipartite graph. Prove that $\mathrm{Jac}(K_{m,n}) = \mathbb{Z}_m^{n-2} \oplus \mathbb{Z}_n^{m-2} \oplus \mathbb{Z}_{mn}$.

4.6.14. Let $L(4,4) = K_4 \times K_4$ be the 4×4 lattice graph. Prove that $\mathrm{Jac}(L(4,4)) = \mathbb{Z}_8^5 \oplus \mathbb{Z}_{32}^4$.

4.6.15. Let Shr be the Shrikhande graph. Prove that $\mathrm{Jac}(\mathrm{Shr}) = \mathbb{Z}_2 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{32}^4$.

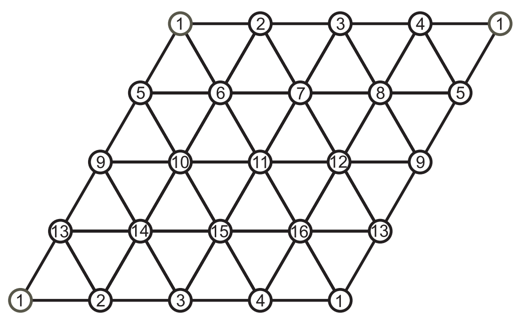

FIGURE 1. Shrikhande graph on the torus

4.6.10. Construct the Abel-Jacobi map of X into $\mathrm{Jac}(X)$ for the following graphs.

(i) $X = C_n$;
(ii) $X = K_4$;
(iii) $X = W_n$, $n = 4, 5$;
(iv) $X = Q_3$.

4.6.11. Let X be a finite connected graph. Denote by $\overline{X}$ the graph obtained from X by collapsing all bridges of X to vertices. Prove $\mathrm{Jac}(X) = \mathrm{Jac}(\overline{X})$

4.6.12. Let e be an edge of graph X such that $X \setminus e = X_1 \cup X_2$ is a disjoint union of two connected graphs X_1 and X_2. Prove that $\mathrm{Jac}(X) = \mathrm{Jac}(X_1) \oplus \mathrm{Jac}(X_2)$.

4.6.13. Let X_1 and X_2 be connected graphs sharing a common vertex. Show that $\mathrm{Jac}(X_1 \dot{+} X_2) = \mathrm{Jac}(X_1 \oplus X_2)$.

4.6.14. Let $A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_r}$ be a finite Abelian group and $X = C_{n_1} \underset{\cdot}{+} C_{n_2} \underset{\cdot}{+} \ldots \underset{\cdot}{+} C_{n_r}$. Show that $\mathrm{Jac}(X) \cong A$.

4.6.15. Find $\mathrm{Jac}(X_1 \times X_2)$ of the Cartesian product $X_1 \times X_2$ of the graphs X_1 and X_2. (unsolved problem)

4.6.16. Prove the isomorphism $\mathrm{Jac}(X) \cong H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$, where $H^1(X, K)$ is the first cohomology group of X over K.

4.6.17. Let $L(X)$ be a Laplacian matrix of a graph X. Prove that $\mathrm{Jac}(X) \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_r}$, where $diag(n_1, n_2, \ldots, n_r, 0)$ is the Smith normal form of $L(X)$.

4.6.18. Are there two graphs X_1 and X_2 with the same Laplacian spectrum whose Jacobians are not isomorphic? Hint: consider two strongly regular graphs with parameters (28, 12, 6, 4) the Lattice graph $L(4,4) = K_4 \times K_4$ and the Shrikhande graph Shr.

4.6.19. Let X and X^* be dual planar graphs. Prove that $\mathrm{Jac}(X) = \mathrm{Jac}(X^*)$.

4.6.20.

4.6.21. [14] Show that the Jacobian of the graph $K_m \times P_n$ is $\mathbb{Z}_t \oplus \mathbb{Z}_{mt}^{m-2}$, where $t = \frac{x_1^n - x_2^n}{\sqrt{m^2+4m}}$, and x_1, x_2 are the two roots of the quadratic equation $x^2 - (m+2)x + 1 = 0 : x_1 = m + 2 + \sqrt{m^2+4m}$, $x_2 = m + 2 - \sqrt{m^2+4m}$.

4.7. **Solutions.**

4.6.18. Consider two graphs on Fig. 4.7. They share the Laplacian polynomial

$$-384x + 1520x^2 - 2288x^3 + 1715x^4 - 708x^5 + 164x^6 - 20x^7 + x^8.$$

In the same time, Jacobian for the first graph is $\mathbb{Z}_{48}$ and for the second $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$.

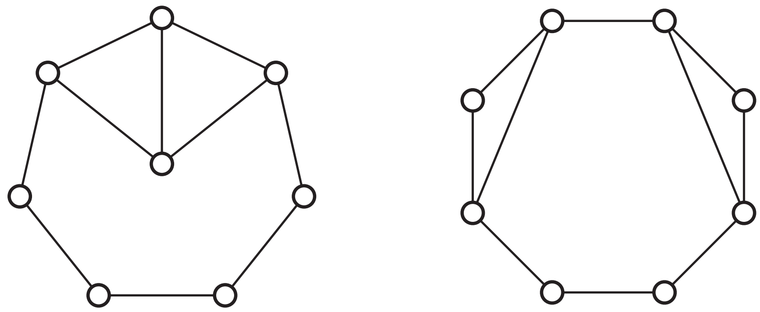

FIGURE 2. Two genus three isospectral graphs

5. Graph of groups and Bass uniformisation theory

6. Riemann-Hurwitz formula and its applications

6.1. **Exersises.**

6.1.1. Let G be a finite group acting on the set of directed edges of a graph X of genus g free and without edge revising. Denote by g' genus of the factor graph $X' = X/G$. Prove that

$$g - 1 = |G|(g' - 1) + \sum_{x \in V(X)} (|G^x| - 1),$$

where $V(X)$ is the set of vertices of X ([3], [8]).

6.1.2. Let X be a graph of genus g and G is a finite group acting on X without edge revising. Denote by $g(X/G)$ genus of the factor graph X/G. Then

$$g - 1 = |G|(g(X/G) - 1) + \sum_{v \in V(X)} (|G^v| - 1) - \sum_{e \in E(X)} (|G^e| - 1),$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^x stands for the stabiliser of $x \in V(X) \cup E(X)$ in G and $|G^x|$ is the order of a stabiliser. (A. D. Mednykh, 2013).

6.1.3. Let X be a graph of genus g and G is a finite group acting on X, possibly with edge revising. Denote by $g(X/G)_{loop}$ genus of the factor graph $(X/G)_{loop}$. Then

$$g - 1 = |G|(g(X/G)_{loop} - 1) + \sum_{v \in V(X)} (|G^v| - 1) - \sum_{e \in E(X)} (|G^e| - 1),$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^x stands for the stabiliser of $x \in V(X) \cup E(X)$ in G and $|G^x|$ is the order of a stabiliser. (A. D. Mednykh, 2013).

6.1.4. Let X be a graph of genus g and G is a finite group acting on X, possibly with edge revising. Denote by $g(X/G)_{tail}$ genus of the factor graph $(X/G)_{tail}$. Then

$$g-1 = |G|(g(X/G)_{tail}-1)+\sum_{v\in V(X)}(|G^v|-1)-\sum_{e\in E(X)}(|G^e|-1)+\sum_{e\in E^{inv}(X)}|G^e|,$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^x is the stabiliser of $x \in V(X) \cup E(X)$ in G, and $E^{inv}(X)$ is the set of invertibile edges of X. (A. D. Mednykh, 2013).

6.1.5. Let X be a graph of genus g and G is a finite group acting on X, possibly with edge revising. Denote by $g(X/G)_{free}$ genus of the factor graph $(X/G)_{free}$. Then

$$g-1 = |G|(g(X/G)_{free}-1)+\sum_{v\in V(X)}(|G^v|-1)-\sum_{e\in E(X)}(|G^e|-1)+\sum_{e\in E^{inv}(X)}|G^e|,$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^x is the stabiliser of $x \in V(X) \cup E(X)$ in G, and $E^{inv}(X)$ is the set of invertible edges of X. (A. D. Mednykh, 2013).

6.1.6. Let X be a graph of genus g and G is a finite group acting on X harmonically, possibly with edge revising. Denote by $g(X/G)_{free}$ genus of the factor graph $(X/G)_{free}$. Then

$$g-1 = |G|(g(X/G)_{free}-1)+\sum_{v\in V(X)}(|G^v|-1)+|E^{inv}(X)|,$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^v is the stabiliser of $v \in V(X)$ in G, and $E^{inv}(X)$ is the set of invertible edges of X [3].

6.1.7. Denote by $M(g)$ maximum size of a finite group acting harmonically on a graph of genus $g \geq 2$. Prove the following result by Scott Corry [8]. For any $g \geq 2$ we have

$$4(g-1) \leq M(g) \leq 6(g-1).$$

The upper and lower bounds are attained for infinitely many values of g.

6.1.8. Let X be a graph of genus g and S is a subset of vertices of X consisting of $s \geq 1$ elements. Suppose that $g-1+s>0$ and G is a finite group acting on X harmonically and leaving the set S invariant. Then

$$|G| \leq 2(g-1)+2s.$$

(R. Nedela, A. D. Mednykh, 2013).

6.1.9. Prove the following discrete version of the first Arakawa's theorem.

Let X be a graph of genus $g \geq 2$ and A and B are two disjoint subsets of vertices of X of the orders $|A| \geq |B| \geq 1$. Suppose that G is a finite group acting harmonically on X and leaving the sets A and B invariant. Then

$$|G| \leq \frac{3(g-1)+|A|+3|B|}{2}.$$

The upper bound is sharp and is attained for arbitrary large values of g. (R. Nedela, A. D. Mednykh, I. A. Mednykh, 2013).

6.1.8. Prove the following discrete version of the second Arakawa's theorem.

Let X be a graph of genus $g \geq 2$ and A, B and C are three disjoint subsets of vertices of X of the orders $|A| \geq |B| \geq |C| \geq 1$. Suppose that G is a finite group acting harmonically on X and leaving the sets A, B and C invariant. Then

$$|G| \leq \frac{g-1+|A|+|B|+|C|}{2}.$$

The upper bound is sharp and is attained for arbitrary large values of g. (R. Nedela, A. D. Mednykh, I. A. Mednykh, 2013).

6.1.9. Prove the following discrete version of the Wiman's theorem.

Let X be a graph of genus $g \geq 2$ and $\mathbb{Z}_N$ is a cyclic group acting harmonically on X. Then $N \leq 2g+2$. The upper bound $N=2g+2$ is attained for any even g. In this case, the signature of orbifold $X/\mathbb{Z}_N$ is $(0;2,g+1)$, that is $X/\mathbb{Z}_N$ is a tree with two branch points of order 2 and $g+1$ respectively. (A. D. Mednykh, I. A. Mednykh, 2013).

7. Miscellaneous questions of graph theory

7.1. **Exersises.**

7.1.1. Prove the following discrete version of R. D. M. Accola formula.

Let X be a finite graph of genus g. Suppose X admits a finite group of harmonic automorphisms, G_0, where $G_0 = \cup_{i=1}^{s} G_i$, $G_i \cap G_j = \langle 1 \rangle$, i, $j > 0$ is a group with a partition. Let the order of G_i be n_i, let $X_i = X/G_i$, and let g_i, be the genus of X_i, for $i = 0, 1, \ldots, s$. Then

$$(s-1)g + n_0 g_0 = \sum_{i=1}^{s} n_i g_i.$$

7.1.2. Let $\mathbb{D}_n = \langle R, V; R^n = V^2 = (RV)^2 = 1 \rangle$ be the dihedral group of order $2n$. Suppose that $\mathbb{D}_n$ acts harmonically on a finite graph X. Denote by $g(X)$ the genus of X. Then

$$g(X) + 2g(X/\mathbb{D}_n) = g(X/\langle R \rangle) + g(X/\langle V \rangle) + g(X/\langle RV \rangle).$$

7.1.3. (Uniqueness of hyperelliptic involution). Let X be a graph of genus $g \geq 2$ and τ_i, $i = 1, 2$ are hyperelliptic involutions on X. That is τ_1 and τ_2 act harmonically on X and the factor-graphs $X/\langle \tau_1 \rangle$ and $X/\langle \tau_2 \rangle$ are trees. Prove that $\tau_1 = \tau_2$.

How many hyperelliptic involutions admit a tree and a flower? By definition, a tree and and a flower are graphs of genera zero and one respectively.

7.1.4. (Uniqueness of γ-hyperelliptic involution). Let γ be a non-negative integer. Let X be a graph of genus g so that $g > 4\gamma + 1$. Suppose τ is an automorphism of X of order two so that the genus of $X/\langle \tau \rangle$ is g. Prove that these properties define τ uniquely and $\langle \tau \rangle$ is central in the full group of automorphisms of X.

7.1.5 Let g_i and g_2 be nonnegative integers. Let X be a graph of genus g so that $2g \geq 3g_1 + 3g_2 + 3$. Let X admits two distinct automorphisms A_1 and A_2, both of period two so that the genus of $X/\langle A_i \rangle$ is g_i. Then, A_1 and A_2 commute.

7.2. **Solutions.**

7.1.1 For the coverings $X \to X/G_0$ and $X \to X/G_i$ the Riemann-Hurwitz formula gives

(1) $$g - 1 = n_0(g_0 - 1) + r_0 \text{ and } g - 1 = n_i(g_i - 1) + r_i,$$

where r_0, r_i are the ramifications of the coverings under consideration. Let $v \in V(X)$ be a branch point of the covering $X \to X/G_0$. For any subgroup H of the group G_0 denote by H^v the stabiliser of v in H and by $|H^v|$ the order of this stabiliser. Then the contribution of v to the ramifications r_0 and r_i is given by $|G_0^v| - 1$ and $|G_i^v| - 1$, respectively. More precisely, we have

$$r_0 = \sum_{v \in V(X)} (|G_0^v| - 1) \text{ and } r_i = \sum_{v \in V(X)} (|G_i^v| - 1). \tag{2}$$

Since G_0 is a group with the partition $\{G_1, G_2, \ldots, G_s\}$ we obtain

$$|G_0| - 1 = \sum_{1 \le i \le s} (|G_i| - 1),$$

or

$$n_0 - 1 = \sum_{1 \le i \le s} (n_i - 1). \tag{3}$$

In a similar way, since the stabiliser G_0^v is a group with the partition $\{G_1^v, G_2^v, \ldots, G_s^v\}$ we have

$$|G_0^v| - 1 = \sum_{1 \le i \le s} (|G_i^v| - 1).$$

Summing the latter equality through all $v \in V(X)$, from (2) we obtain

$$r_0 = \sum_{1 \le i \le s} r_i. \tag{4}$$

Now substitute equations (1) into (4).

$$(g - 1) - n_0(g_0 - 1) \;=\; \sum_{1 \le i \le s} [(g - 1) - n_i(g_i - 1)]. \tag{5}$$

Substracting (3) from (5) we obtain the result.

7.1.2 By the Accola's formula (7.1.1) we have

$$ng(X) + 2ng(X/\mathbb{D}_n) = ng(X/\langle R \rangle) + 2 \sum_{i=0}^{n-1} g(X/\langle R^i V \rangle).$$

If n is odd, then all subgroups $\langle R^i V \rangle$ are conjugate, and hence $g(X/\langle R^i V \rangle) = g(X/\langle V \rangle)$ for $i = 1, 2, \ldots, n-1$.

If n is even, then $\langle R^i V \rangle)$ and $\langle R^j V \rangle)$ are conjugate if and only $i \equiv$ mod 2. Thus we have $\sum\limits_{i=0}^{n-1} g(X/\langle R^i V \rangle)$ is equal to $ng(X/\langle V \rangle)$ for odd n, and $\frac{n}{2}(g(X/\langle V \rangle) + g(X/\langle R V \rangle))$ for even n. Therefore we obtain the desired result.

7.1.3 See the proof of (7.1.4) for the case $\gamma = 0$.

7.1.4 Suppose τ_1 and τ_2 are two distinct automorphisms of X with the properties of τ. Then, τ_1 and τ_2 generate a dihedral group, $\mathbb{D}_n$, of order $2n$. We set $R = \tau_1\tau_2$ and $V = \tau_2$. Then the Accola's formula gives

$$g + 2g(X/D_n) = 2g + g(X/\langle R\rangle).$$

From the Riemann-Hurwitz formula for graphs applied to $X \to X/\langle R\rangle$ we have

$$g - 1 = n(g(X/\langle R\rangle) - 1) + r,$$

where n is the order of R. But $n \geq 2$ since τ_1 and τ_2 are distinct and $r > 0$. So

$$g - 1 \geq 2(g(X/\langle R\rangle) - 1) \text{ or } 2g(X/\langle R\rangle) \leq g + 1.$$

Since $g(X/\mathbb{D}_n) \geq 0$ we have

$$2g \leq 2g + 4g(X/D_n) = 4\gamma + 2g(X/\langle R\rangle) \leq 4\gamma + g + 1,$$

or

$$g \leq 4\gamma + 1.$$

This contradiction shows that τ is unique.

Let t be another automorphism of X. Then, $t\tau t^{-1}$ has the same properties as τ. Thus $\tau = t\tau t^{-1}$, and the proof is complete.

7.1.5 Setting $g_0 = g(X/\langle A_1, A_2\rangle)$ and $g_3 = g(X/\langle A_1A_2\rangle)$ by the Accola's formula we have

$$g + 2g_0 = g_1 + g_2 + g_3.$$

Let the product A_1A_2 have order n. We wish to show that n is two, so suppose $n \geq 3$. The Riemann-Hurwitz formula for $X \to X/\langle A_1A_2\rangle$ is

$$g - 1 = n(g_0 - 1) + r.$$

Hence, $g - 1 \geq 3(g_0 - 1)$ or $3g_0 \leq g + 2$.

Since, $g_0 \geq 0$ we have

$$3g \leq 3g + 6g_0 = 3g_1 + 3g_2 + 3g_3$$

or

$$3g \leq 3g_1 + 3g_2 + 2 + g.$$

This contradicts the hypothesis and thus $n = 2$. Then $\langle A_1, A_2\rangle$ is the dihedral group $\mathbb{D}_2$ of order four and the elements A_1 and A_2 commute.

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